

# Detection of Arbitrage Opportunities and Forecasting in the Foreign Exchange Markets

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## Abstract:

In this paper, the existence of no arbitrage assumption over time and triangular arbitrage in a foreign exchange market is tested. Also, forecasting the exchange rates are studied. An optimal property of exchange rates returns to guarantee the no arbitrage assumption as well as for forecasting exchange rates is the martingale property. Some theoretical results under the risk neutral measure and their equivalents form in physical probability measure are given. Also, based on a real data set, it is seen that this assumption works well for forecasting purposes. Using the Doob maximal inequality, the accuracy of forecasts is surveyed. Then, a theoretical relation between beta market risks of exchange rates is surveyed. Finally, a conclusion section is also proposed.

**Keywords:** arbitrage opportunities; CAPM; Doob maximal inequality; forecasting; foreign exchange markets; market risk; martingale; triangular arbitrage.

**JEL Classification:** G 21.

## Introduction

In this paper, first, the existence of arbitrage opportunity in a foreign exchange market is surveyed. Then, forecasting exchange rate problem is studied. About the arbitrage opportunity, it is famous that existence of a unique risk neutral probability measure is equivalent the market is arbitrage free and complete. A result of the first fundamental asset pricing theorem is that under the risk neutral measure, each discounted share prices in market is martingale, Bjork (2009). To apply this idea in a foreign exchange market, let  $x_t$  and  $y_t$  be two exchange rates of three currencies at time  $t$ . For example, suppose that they are EUR/USD and USD/GBP exchange rates, respectively. It is known that the product of  $x_t$  and  $y_t$ , itself, is another exchange rate. Here,  $x_t y_t = z_t$  is EUR/GBP exchange rate. For each fixed  $t$ , to remove the triangular arbitrage opportunity, it is necessary to have  $x_t y_t = z_t$ . Ma (2008) using this equation, applied a matrix approach for identifying foreign exchange arbitrage opportunities.

Hau (2014) investigated the arbitrage opportunities in foreign exchange markets involving multi-currencies exchange rates by considering liquidity level of each currency. Also, the discounted format of  $x_t$ ,  $y_t$  and  $z_t$  should be martingale with respect to natural filtrations  $F_t^x$ ,  $F_t^y$  and  $F_t^z$  where  $F_t^A$  which is  $\sigma$ -field generated by  $\{A_s, s \leq t\}$  (for  $= x, y, z$ ). Foreign exchange market should be arbitrage free from both sights. Thus, a natural question that arises is under which conditions  $e^{-rt} A_t$  (for  $= x, y, z$ ) is martingale with respect to the product of natural filtrations that is  $G_t = F_t^x \times F_t^y \times F_t^z$  as well as the condition  $x_t y_t = z_t$  holds? In the first part of the current paper, this question is answered.

The second problem of the current paper is the concept of forecasting the future exchange rates. It is well-known that there are four famous ways to forecast the exchange rates including purchasing power parity (PPP), relative economic strength, time series model, and bottom line approaches (see Hao 2014). In this paper, a method for forecasting the exchange rates base on the martingale property is studied. The rest of paper is organized as follows. Section 2 presents main theoretical results of problem about the no arbitrage assumption under the risk neutral measures as well as forecasting the exchanges rates under physical measures. Conclusions are given in Section 3.

## 1. Research Results

This section has two parts. The first part studies the no arbitrage assumption and the second considers the forecasting problem in a specified foreign exchange market.

## 1.1. No Arbitrage Model

The basic idea of this paper backs to exercise 1.15.9 of Calin (2012, 31). Chauvin (1991) considered the same problems in branching Brownian motions. This problem is also studied by Cherny (2011) in eight formulations, for examples in independent martingales, local martingales, continuous martingales and etc. Tian (2011) also studied this problem with applications to logrank test and Cox's model. Most of literatures considers the independent martingales, by the way, in our case, exchange rates  $x_t$ ,  $y_t$  and  $z_t$  are correlated which may be modeled by copula function. The same problem proposes for quadrangular and multiple currencies arbitrage cases, however, in this note, only the triangular cases is studied.

Suppose that, under the two risk neutral measure, then,  $x_t$  and  $y_t$  follow the following the Black-Scholes equations

$$\begin{cases} dx_t = rx_t dt + \sigma_1 x_t dB_1 \\ dy_t = ry_t dt + \sigma_2 y_t dB_2, \end{cases}$$

where:  $B_1$  and  $B_2$  are two correlated Brownian motions such that  $\text{cor}(dB_1, dB_2) = \rho$  and  $r$  is the risk free rate.

Let  $dB_2 = \rho dB_1 + \sqrt{1 - \rho^2} dB_3$ , such that  $B_1$  and  $B_3$  are two independent Brownian motions. Substituting this equation in  $dy_t$  and applying the Ito lemma for  $z = f(x, y) = xy$ , it is seen that:

$$dz = (2r + \rho\sigma_1\sigma_2)zdt + (\sigma_1 + \sigma_2\rho)zdB_1 + \sigma_2\sqrt{1 - \rho^2}zdB_3.$$

Define  $U = e^{-rt}z$ . Then,

$$dU = (r + \rho\sigma_1\sigma_2)zdt + (\sigma_1 + \sigma_2\rho)zdB_1 + \sigma_2\sqrt{1 - \rho^2}zdB_3.$$

Therefore,  $dU$  hasn't the term  $dt$  (a necessary and sufficient condition for a Black-Scholes equation to be martingale) if and only if:  $r + \rho\sigma_1\sigma_2 = 0$ . Equivalently,

$$\text{cov}(dB_1, dB_2) = -r dt.$$

The following proposition summarizes the above discussion.

*Proposition 1.* There is no arbitrage in a foreign exchange market if and only if, under the risk neutral measure, for any two exchange rates  $x_t$  and  $y_t$ , then  $\text{cov}\left(\frac{dx_t}{x_t}, \frac{dy_t}{y_t}\right) = -r dt$  or, equivalently, then  $\text{cor}\left(\frac{dx_t}{x_t}, \frac{dy_t}{y_t}\right) = -r/\sigma_1\sigma_2$ .

*Remark 1.* A natural question is: under which conditions the  $U$  process is martingale under the actual probability measure? One can see that the necessary and sufficient condition for  $U$  being the martingale under the actual measures is that

$$\mu_1 + \mu_2 - r + \rho\sigma_1\sigma_2 = 0.$$

*Remark 2.* Here, the validity of above proposition in discrete time process is surveyed. Let:

$$x_t = x_0 \prod_{i=1}^t (1 + r_i) \text{ and } y_t = y_0 \prod_{i=1}^t (1 + s_i),$$

where:  $r_t$  and  $s_t$  are the daily return of exchange rates of  $x_t$  and  $y_t$ . It is easy to see that:

$$x_t \approx x_0 \exp\left(\sum_{i=1}^t r_i\right) \quad \text{and} \quad y_t \approx y_0 \exp\left(\sum_{i=1}^t s_i\right). \text{ Then, } z_t \approx z_0 \exp\left(\sum_{i=1}^t (r_i + s_i)\right).$$

Let  $F_t$  be  $\sigma$ -field generated by  $\{(r_i, s_i), i = 1, \dots, t\}$ . Assuming  $\{(r_i, s_i), i > 0\}$  be a sequence of independent variables, then:

$$E(e^{-rt}z_t | F_s) = e^{-rs}z_s E\left(e^{r(s-t) + \sum_{i=s+1}^t (r_i + s_i)}\right).$$

Suppose that, under the risk neutral probability measure, the mean of  $r_i, s_i$  be the risk free rate and standard deviation  $\sigma_i$ ,  $i = 1, 2$ . Next suppose that  $r_i, s_i$  are jointly normal distribution with correlation  $\rho$ . Therefore,  $\sum_{i=s+1}^t (r_i + s_i)$  has normal distribution with mean  $2r(t - s)$  and variance  $(t - s)\{2\rho\sigma_1\sigma_2\}$ . Thus, it is seen that:

$$E\left(e^{r(s-t) + \sum_{i=s+1}^t (r_i + s_i)}\right) = e^{(t-s)\{r + \rho\sigma_1\sigma_2\}}.$$

The above expectation is one if and only if:  $\text{cov}(r_i, s_i) = -r$ . The proposition 1 studied the covariance between returns of exchange rates. The following proposition studies the covariance structure between exchange rates, themselves.

*Proposition 2.* Under the risk neutral measure, then:  $\text{cov}(x_t, y_t | G_s) = e^{r(t-s)} [1 - e^{r(t-s)}] z_s$ .

*Proof.* Notice that:

$$\begin{cases} E(e^{-rt} x_t | G_s) = e^{-rs} x_s \\ E(e^{-rt} y_t | G_s) = e^{-rs} y_s \\ E(e^{-rt} x_t y_t | G_s) = e^{-rs} x_s y_s, \end{cases}$$

This completes the proof.

The following proposition gives recursive relations to compute the covariance structure between two exchange rates, sequentially.

*Proposition 3.* Let:

$\pi_t = \text{cov}(x_t, y_t)$ ,  $E(x_t) = \mu_t$  and  $E(y_t) = \theta_t$ . Then,

$$\pi_t = e^{r(t-s)} \pi_s + e^{r(t-s)} (1 - e^{r(t-s)}) \mu_s \theta_s.$$

Again, it is seen that the proposition 3 is correct in the case of discrete time processes. This fact is surveyed in the remark 3.

*Remark 3.* Let  $s = t - 1$ , in discrete time case. Then,

$$\pi_t = e^r \pi_{t-1} + e^r (1 - e^{r(t-s)}) \mu_{t-1} \theta_{t-1}.$$

Solving this equation recursively, it is seen that:

$$\pi_t = e^{r(t-1)} \pi_1 + (1 - e^r) \sum_{j=1}^{t-1} e^{r(t-j)} \mu_j \theta_j.$$

Notice that  $\mu_j = x_0 e^{rj}$  and  $\theta_j = y_0 e^{rj}$ , then:

$$\pi_t = e^{r(t-1)} \pi_1 - x_0 y_0 e^{r(t+1)} (e^{r(t-1)} - 1).$$

Ma (2008) identified the arbitrage opportunities in a foreign exchange market using the maximum eigenvalue of exchange rate matrix. Suppose that market contains  $n$  currencies  $C_i$ ,  $i = 1, 2, \dots, n$ . Let  $x_t^i$  for  $i = 1, 2, \dots, n$  be the exchange rate of  $C_i$  with respect to  $C_{i+1}$  at time  $t$ , (assuming  $C_{n+1} = C_1$ ). Let  $\lambda_t^i$  be the  $i$ -th eigenvalue, then:

$$\lambda_t^{\max} = \max(\lambda_t^1, \dots, \lambda_t^n),$$

subject to  $\sum_{i=1}^n \lambda_t^i = n$ . This defines a multi-criteria decision making (MCDM) problem. An equivalent weighting approach is to solve:

$$\lambda_t^{\max} = \max \Delta = \sum_{i=1}^n w_t^i \lambda_t^i,$$

subject to  $\sum_{i=1}^n \lambda_t^i = n$ .

Saaty (1980) proved that when there is no arbitrage opportunity, then  $\lambda_t^{\max} = n$  (other eigenvalues are zero). In case of existence arbitrage  $\lambda_t^{\max} > n$  and other eigenvalues are negative. Here, a martingale property of  $\lambda_t^{\max}$  is proved as follows. Let  $A_t$  be the exchange rate matrix at time  $t$ . It is easy to see that  $e^{-rt} A_t$  is martingale with respect to  $G_t$ , under the risk neutral measure. From the definition of eigenvalue the  $e^{-rt} \lambda_t^{\max}$  is martingale with respect to  $G_t$ . The following proposition summarizes this result.

*Proposition 4.* (a) and (b) are true.

(a) The  $e^{-rt} \lambda_t^{\max}$  is martingale with respect to  $G_t$ , under the risk neutral measure, and

(b) Let  $L_t$  be the derivative of risk neutral probability measure with respect to physical probability measure given  $G_t$ . Then,  $e^{-rt} L_t \lambda_t^{\max}$  is martingale with respect to  $G_t$  under physical probability measure.

## 1.2. Forecasting

Here, it is interested to study the martingale property under the physical probability measures. It helps to forecast the exchange rates. Let  $\mu_i$ ,  $i = 1, 2, 3$  be the expected rate of returns of exchange rates of  $x$ ,  $y$  and  $z$ , respectively.

*Proposition 5.* Let  $x'_t = e^{-\mu_1 t} x_t$  and  $y'_t = e^{-\mu_2 t} y_t$ . Then,  $x'_t$  and  $y'_t$  are martingale with respect to physical probability measures. Then,  $e^{-t\{\rho\sigma_1\sigma_2\}} x'_t y'_t$  is also martingale.

*Proof.* One can see that:  $e^{-(\mu_1 + \mu_2 + \rho\sigma_1\sigma_2)t} z_t = e^{-t\{\rho\sigma_1\sigma_2\}} x'_t y'_t$ . Is martingale with respect to the physical (actual) probability measure. This completes the proof.

*Remark 4.* Consider the case of  $\mu_1 + \mu_2 = 0$ . Then,  $e^{-(\rho\sigma_1\sigma_2)t} z_t$  is martingale. Then,

$$E(z_T | \sigma(z_s; s \leq t)) = e^{-(T-t)\{\rho\sigma_1\sigma_2\}} z_t.$$

when:  $\rho \rightarrow 0$ , then  $z_t$  is also a martingale.

Here, using the Doob maximal inequality (see Bjork 2009), results about the accuracy of forecasts are proposed. Notice that assuming  $\mu_1 + \mu_2 + \rho\sigma_1\sigma_2$  is positive, then, for  $t \leq T$ ,  $e^{-(\mu_1 + \mu_2 + \rho\sigma_1\sigma_2)T} z_t \leq e^{-(\mu_1 + \mu_2 + \rho\sigma_1\sigma_2)t} z_t$ . Therefore,

$$P(\sup_{0 \leq t \leq T} e^{-(\mu_1 + \mu_2 + \rho\sigma_1\sigma_2)T} z_t > \alpha) \leq P(\sup_{0 \leq t \leq T} e^{-(\mu_1 + \mu_2 + \rho\sigma_1\sigma_2)t} z_t > \alpha).$$

According to the Doob maximal inequality, the second term is less or equal than:

$$\frac{E(e^{-2(\mu_1 + \mu_2 + \rho\sigma_1\sigma_2)T} z_T^2)}{\alpha^2}.$$

$$\text{Thus, } P(\sup_{0 \leq t \leq T} e^{-(\mu_1 + \mu_2 + \rho\sigma_1\sigma_2)T} z_t > \alpha) = P(e^{-(\mu_1 + \mu_2 + \rho\sigma_1\sigma_2)T} \sup_{0 \leq t \leq T} z_t > \alpha) \leq \frac{e^{-2(\mu_1 + \mu_2 + \rho\sigma_1\sigma_2)T} E(z_T^2)}{\alpha^2}.$$

$$\text{Then, } P(\sup_{0 \leq t \leq T} z_t > \pi) \leq \frac{E(z_T^2)}{\pi^2}.$$

The following proposition summarizes the above discussions.

*Proposition 6.* For every threshold  $\pi$ , assuming  $\mu_1 + \mu_2 + \rho\sigma_1\sigma_2 > 0$ , then

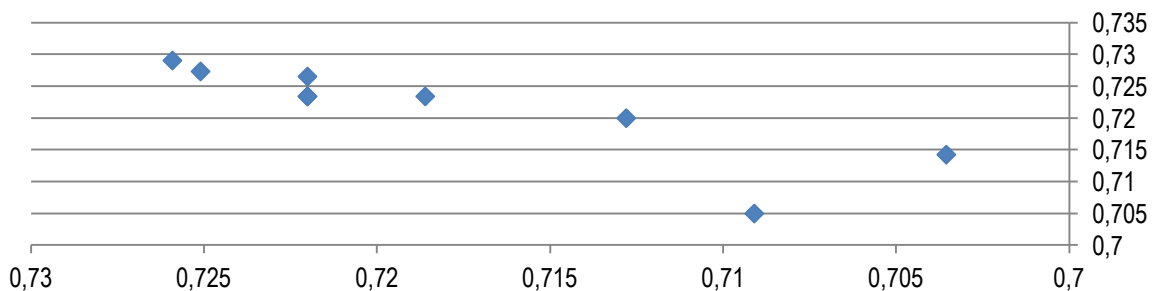
$$P(\sup_{0 \leq t \leq T} z_t > \pi) \leq \frac{E(z_T^2)}{\pi^2}.$$

If  $\mu_1 + \mu_2 + \rho\sigma_1\sigma_2 < 0$ , then

$$P(\sup_{0 \leq t \leq T} z_t > \pi) \geq \frac{E(z_T^2)}{\pi^2}.$$

*Example 1.* Consider the EUR/USD ( $x$ ) and USD/GBP ( $y$ ) exchange rates. Then,  $z$  is the EUR/GBP rate. The data set consists the daily exchange rates between 26/02/2015 to 12/03/2015. It is seen that  $\mu_1 + \mu_2 = -0.00192$ . Also,  $\sigma_1 = 0.005225$ ,  $\sigma_2 = 0.00192$  and  $\rho = -0.4538$ . The following plot shows the forecasts of EUR/GBP rates for 04/03/2015 to the end of period study versus the actual rates. It is seen that the forecasts are good.

Figure 1. Forecasts vs actual EUR/GBP rates



In this part of paper, it is interested to study the market risk of exchange rates of  $x$ ,  $y$ , and  $z$ . To this end, suppose that the Capital Asset Pricing Model (CAPM) satisfies. That is,

$$\mu_i = r + \beta_i(E(r_m) - r), i = 1, 2, 3,$$

where:  $\beta_i$  is the market risk of  $i$ -th exchange rate and  $E(r_m)$  is the expected rate of return of market of foreign exchange. Then, the following proposition is given.

*Proposition 7.* (a) and (b) are true.

(a) Parameter  $\mu_3$ , the expected rate of return of exchange rate  $z$ , can be written as:

$$\mu_3 = \mu_1 + \mu_2 + \rho\sigma_1\sigma_2.$$

(b) Substituting the CAPM model in the above formula for  $\mu_3$ , then:

$$\beta_3 = \beta_1 + \beta_2 + \frac{r + \rho\sigma_1\sigma_2}{E(r_m) - r}.$$

The above relation gives an equation between market risk of foreign exchanges  $x$ ,  $y$  and  $z$ .

## Conclusions

This short note is concerned about two main problems in a specified foreign exchange market. The first problem is the detection of simultaneous triangular arbitrage and arbitrage over time in a foreign exchange market. The martingale property of some indicators like the maximum eigenvalues of exchange rate matrix is proved and suggested to detect the arbitrage opportunities. The same results may be presented in quadrangular and multiple currencies arbitrage cases, however, these extensions are straightforward and hence they are omitted. The second problem is the forecasting exchange rate. Again, using the martingale property, a forecast is proposed and using the maximal inequalities like Doob inequality, the accuracy of forecasts is surveyed. A theoretical relation between beta market risks of exchange rates is given.

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